

ON SOME SPECIAL CLASSES OF COMPLEX ELLIPTIC CURVES

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ABSTRACT. In this paper we classify the complex elliptic curves E for which there exist cyclic subgroups $C \leq (E, +)$ of order n such that the elliptic curves E and E/C are isomorphic, where n is a positive integer. Important examples are provided in the last section. Moreover, we answer the following question: given a complex elliptic curve E , when can one find a cyclic subgroup C of order n of $(E, +)$ such that $(E, C) \sim (\frac{E}{C}, \frac{E[n]}{C})$, $E[n]$ being the n -torsion subgroup of E , classifying in this way the fixed points of the action of the Fricke involution on the open modular curves $Y_0(n)$.

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1. INTRODUCTION

Let E be an elliptic curve defined over the field of complex numbers \mathbb{C} and C be a subgroup (not necessarily cyclic) of order $n < \infty$ of $(E, +)$. This means that C is a subgroup of order n of $E[n]$ where by $E[n]$ one denotes the n -torsion subgroup of E i.e. the set of points of order n in E : $E[n] = \{P \in E : [n]P = O\}$. Let also $\pi : E \rightarrow E/C$ be the natural projection. Since C acts effectively and properly discontinuous on E , the group E/C has a structure of Riemann variety, compatible with the morphism π and moreover π is unramified of degree n : $\deg \pi = |\pi^{-1}(O)| = |C| = n$. It is known that E/C is a complex elliptic curve and that if C is cyclic, one has $E[n]/C \cong \mathbb{Z}/n\mathbb{Z}$.

Denote by \mathcal{H} the upper half plane i.e. $\{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$. One defines the open modular curves $Y_0(n)$ as the quotient space $\Gamma_0(n)/\mathcal{H}$ that is the set of orbits $\{\Gamma_0(n)\tau : \tau \in \mathcal{H}\}$, where $\Gamma_0(n)$ is the "Nebentypus" congruence subgroup of level n of $SL_2(\mathbb{Z})$, acting on \mathcal{H} from the left:

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{S}L_2(\mathbb{Z}) \mid c \equiv 0 \pmod{n} \right\}.$$

Following the notations of [DS], an enhanced elliptic curve for $\Gamma_0(n)$ is by definition an ordered pair (E, C) where E is a complex elliptic curve and C a cyclic subgroup of E of order n . Two pairs (E, C) and (E', C') are said to be equivalent if some isomorphism $E \cong E'$ takes C to C' . One denotes the set of equivalence classes with

$$S_0(n) := \{\text{enhanced elliptic curves for } \Gamma_0(n)\} / \sim.$$

Furthermore, an element of $S_0(n)$ is an equivalence class $[E, C]$. $S_0(n)$ is a moduli space of isomorphism classes of complex elliptic curves and n -torsion data.

Denote now by Λ_τ the lattice $\mathbb{Z} + \mathbb{Z}\tau$, $\tau \in \mathcal{H}$ and by E_τ the elliptic curve \mathbb{C}/Λ_τ . Then one has the following bijection:

$$S_0(n) \cong Y_0(n) \text{ given by } [\mathbb{C}/\Lambda_\tau, \langle 1/n + \Lambda_\tau \rangle] \mapsto \Gamma_0(n)\tau \text{ (see [DS], Theorem 1.5.1 for details.)}$$

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In this paper we study the complex elliptic curves E for which there exist cyclic subgroups $C \leq (E, +)$ of order n such that the elliptic curves E and E/C are isomorphic, where n is a positive integer. We observe that E (and consequently E/C) are CM curves, obviously have isomorphic endomorphism rings and hence these points (E, C) with $E \cong E/C$ are a special class of Heegner points on the modular curve $X_0(n) = Y_0(n) \cup \{\text{cusps}\}$. Very nice examples are provided in the last section for the cases $n = 2, 3, 5$ (and we remark that for p prime there are exactly $p+1$ complex elliptic curves E (up to an isomorphism) which admit at least one subgroup $C \leq (E, +)$ of order p such that $\frac{E}{C} \simeq E$). We recall that the Heegner points of $Y_0(n)$ as defined by Birch in [B1] are pairs (E, E') of n -isogenous curves with the same endomorphism ring. Heegner was the one introducing them in [He] while working on the class number problem for imaginary quadratic fields and their importance is extensively described in the survey [B2].

After elementarily studying the above mentioned class of Heegner points, upon imposing certain conditions (see Theorem 2.3 and Proposition 2.4), given a complex elliptic curve E , we answer the question of when there exists $C \leq (E, +)$ cyclic subgroup of order n of E such that $(E, C) \sim (\frac{E}{C}, \frac{E[n]}{C})$, studying in this way the fixed points of the action of the Fricke involution

$$w_n := \begin{pmatrix} 0 & -1 \\ n & 0 \end{pmatrix} \in GL_2(\mathbb{Q}^+)$$

on the open modular curves $Y_0(n)$.

The number of fixed points of the Fricke involution on $Y_0(n)$ was computed by Ogg (see [O], Proposition 3) and Kenku (see [K], Theorem 2) and this number, for $n > 3$, is $\nu(n) = h(-n) + h(-4n)$ if $n \equiv 3 \pmod{4}$ and $\nu(n) = h(-4n)$ otherwise, where $h(-n)$ is the class number of primitive quadratic forms of discriminant $-n$ and $\nu(2) = \nu(3) = 2$.

It is easy to see that σ normalizes the group $\Gamma_0(n)$ and hence gives an automorphism $\Gamma_0(n)z \mapsto \Gamma_0(n)\sigma(z)$ on $Y_0(n)$. It is also easy to check that this automorphism is an involution. The following proposition is a known result which we will use throughout the paper and we skip its proof (the reader may consult [DR], section IV), 4.4 or [Hu], Theorem 2.4 and Remark 5.5 for details):

Proposition 1.1. *The action of w_n on the moduli space $S_0(n)$ is given by: $[E, C] \mapsto [E/C, E[n]/C]$.*

2. MAIN RESULTS

With the background from the previous section we are ready to classify the complex elliptic curves E for which there exist cyclic subgroups $C \leq (E, +)$ of order n such that the elliptic curves E and E/C are isomorphic:

Theorem 2.1. *Let E be a complex elliptic curve determined by the lattice $\langle 1, \tau \rangle$, $\tau \in \mathcal{H}$. Then:*

i) $\exists C \leq (E, +)$ finite cyclic subgroup such that $\frac{E}{C} \simeq E \Leftrightarrow \exists u, v \in \mathbb{Q}$ such that $\tau^2 = u\tau + v$ with $\Delta = u^2 + 4v < 0$ (i.e. E admits complex multiplication);

ii) If τ satisfies the conditions of i) and $u = \frac{u_1}{u_2}, v = \frac{v_1}{v_2}, u_2 \neq 0, v_2 \neq 0, u_1, u_2, v_1, v_2 \in \mathbb{Z}, (u_1, u_2) = (v_1, v_2) = 1, d_2 = (u_2, v_2)$, then:

$\exists C \leq (E, +)$ cyclic subgroup of order n which satisfies $\frac{E}{C} \simeq E \iff \exists (a, b') \in \mathbb{Z}^2$ with $(a, b') = 1$ such that $n = \det M$, where M is the matrix

$$M = \begin{pmatrix} a & A \\ b & B \end{pmatrix}$$

and $(a, A, b, B) = \left(a, \frac{u_2 v_1}{d_2} b', \frac{u_2 v_2}{d_2} b', a + \frac{u_1 v_2}{d_2} b'\right)$;

iii) The subgroup C from ii) is $C = \langle \frac{u_{11} + u_{21}\tau}{n} \rangle$, where u_{11}, u_{21} are obtained in the following way:

Since $\det M = n$ and $\gcd(a, A, b, B) = 1$ (one deduces easily this), the matrix M is arithmetically equivalent with the matrix:

$$M \sim \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix},$$

hence

$$\exists U, V \in GL_2(\mathbb{Z}) \quad \text{such that} \quad M = U \cdot \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \cdot V.$$

The elements u_{11}, u_{21} are the first column of the matrix

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}.$$

Proof. We have that $L = \mathbb{Z} + \mathbb{Z}\tau$ and $C = \langle \bar{g} \rangle \leq E = \frac{\mathbb{C}}{L}$ cyclic subgroup of order n hence $C = \frac{L + \mathbb{Z}g}{L}$ and $n\bar{g} = \bar{0}$. It follows that $ng \in L$ and hence we can write $g \in \mathbb{C}$ as

$$(1) \quad g = \frac{\alpha_1 + \alpha_2 \tau}{n}, \alpha_1, \alpha_2 \in \mathbb{Z}.$$

Since $\text{ord}(\bar{g}) = n$ it follows easily that $\gcd(\alpha_1, \alpha_2, n) = 1$.

We have that $\frac{E}{C} \simeq \frac{\frac{\mathbb{C}}{L}}{\frac{L + \mathbb{Z}g}{L}} \simeq \frac{\mathbb{C}}{L + \mathbb{Z}g} = \frac{\mathbb{C}}{L'}$, where we denoted by $L' = L + \mathbb{Z}g = \mathbb{Z} + \mathbb{Z}\tau + \mathbb{Z}g$.

It is known that $\frac{\mathbb{C}}{L} \simeq \frac{\mathbb{C}}{L'} \iff \exists \lambda \in \mathbb{C}$ such that $\lambda L = L'$. Consequently $E \simeq \frac{E}{C} \iff \frac{\mathbb{C}}{L} \simeq \frac{\mathbb{C}}{L'} \iff \exists \lambda \in \mathbb{C}$ such that

$$(2) \quad \lambda(\mathbb{Z} + \mathbb{Z}\tau) = \mathbb{Z} + \mathbb{Z}\tau + \mathbb{Z}g.$$

The relation (2) can be studied in the following way:

" \subseteq " Since g is given by (1), from the inclusion " \subseteq " one shows that (note that this is not an equivalence):

$\exists a, b, A, B \in \mathbb{Z}$ such that

$$(3) \quad \begin{cases} \lambda = \frac{a+b\tau}{n} \\ \lambda\tau = \frac{A+B\tau}{n} \end{cases}$$

" \supseteq " The inclusion " \supseteq " is equivalent with

$$(4) \quad \exists \alpha, \beta, u, v, s, t \in \mathbb{Z} \quad \text{such that} \quad \begin{cases} 1 = \alpha\lambda + \beta\lambda\tau \\ \tau = u\lambda + v\lambda\tau \\ g = s\lambda + t\lambda\tau \end{cases}$$

By replacing (3) and (1) in (4) we obtain

$$(5) \quad \begin{cases} 1 = \alpha \frac{a+b\tau}{n} + \beta \frac{A+B\tau}{n} \\ \tau = u \frac{a+b\tau}{n} + v \frac{A+B\tau}{n} \\ \frac{\alpha_1 + \alpha_2\tau}{n} = s \frac{a+b\tau}{n} + t \frac{A+B\tau}{n} \end{cases} \iff \begin{cases} \begin{pmatrix} a & A \\ b & B \end{pmatrix} \cdot \begin{pmatrix} \alpha & u \\ \beta & v \end{pmatrix} = nI_2 \\ \begin{pmatrix} a & A \\ b & B \end{pmatrix} \cdot \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \end{cases}$$

□

Note that the relation (2) is not yet equivalent with (3) and (5). In order to obtain the equivalent conditions for (2) let us look at the inclusion " \subseteq ".

The inclusion " \subseteq " means that there exist $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{Z}$ such that

$$(6) \quad \begin{cases} \lambda = a_1 + b_1\tau + c_1g \\ \lambda\tau = a_2 + b_2\tau + c_2g \end{cases} \iff \begin{cases} \frac{a+b\tau}{n} = a_1 + b_1\tau + c_1 \frac{\alpha_1 + \alpha_2\tau}{n} \\ \frac{A+B\tau}{n} = a_2 + b_2\tau + c_2 \frac{\alpha_1 + \alpha_2\tau}{n} \end{cases} \iff$$

$$\begin{cases} a = na_1 + c_1\alpha_1 \\ b = nb_1 + c_1\alpha_2 \end{cases} \quad \text{and} \quad \begin{cases} A = na_2 + c_2\alpha_1 \\ B = nb_2 + c_2\alpha_2 \end{cases} \iff$$

$$(7) \quad \begin{cases} \hat{a} = \hat{c}_1\hat{\alpha}_1 \\ \hat{b} = \hat{c}_1\hat{\alpha}_2 \end{cases} \quad \text{and} \quad \begin{cases} \hat{A} = \hat{c}_2\hat{\alpha}_1 \\ \hat{B} = \hat{c}_2\hat{\alpha}_2 \end{cases} \quad \text{in} \quad \frac{\mathbb{Z}}{n\mathbb{Z}}$$

We obtain that the inclusion " \subseteq " is equivalent with the existence of $c_1, c_2 \in \mathbb{Z}$ which both satisfy (6), i.e. (2) is equivalent with (5) and (6).

We've obtained that the hypothesis $E \simeq \frac{E}{C}$ with C cyclic subgroup of order n is equivalent with the relations (2) and $(\alpha_1, \alpha_2, n) = 1$, i.e. is equivalent with the relations (5), (6), $(\alpha_1, \alpha_2, n) = 1$ and $(a + b\tau)\tau = A + B\tau$. Assume now that these four conditions are satisfied and let us replace them with conditions which are easier to be verified.

Denote by M the matrix

$$M = \begin{pmatrix} a & A \\ b & B \end{pmatrix} \text{ hence } \hat{M} = \begin{pmatrix} \hat{a} & \hat{A} \\ \hat{b} & \hat{B} \end{pmatrix} = \begin{pmatrix} \hat{c}_1\hat{\alpha}_1 & \hat{c}_2\hat{\alpha}_1 \\ \hat{c}_1\hat{\alpha}_2 & \hat{c}_2\hat{\alpha}_2 \end{pmatrix} \text{ in } \frac{\mathbb{Z}}{n\mathbb{Z}} \implies$$

$$\det \hat{M} = \hat{0} \quad \text{in} \quad \frac{\mathbb{Z}}{n\mathbb{Z}} \implies \det M : n$$

By using now (5) we obtain $\det(M) | n^2$ and so $\det(M) = nk, k | n$. From (5) we get $\alpha = \frac{nB}{\det M} = \frac{B}{k}$ hence $k | B$. Similarly one obtains $k | b, k | A, k | a$ and hence from the second equality of (5) we obtain that $k | \alpha_1, \alpha_2$. Since $k | n$ and $(\alpha_1, \alpha_2, n) = 1$ we obtain $k = \pm 1$, consequently

$$\det(M) = \pm n.$$

We will prove that the relations (5), $\gcd(\alpha_1, \alpha_2, n) = 1$ and $\det(M) = \pm n$ imply (6):

In did, from $(\alpha_1, \alpha_2, n) = 1$ it follows that $\exists \mu_1, \mu_2, \mu_3 \in \mathbb{Z}$ such that $\mu_1\alpha_1 + \mu_2\alpha_2 + \mu_3n = 1$.

Choose $\hat{c}_1 = \widehat{\mu_1 a + \mu_2 b}$. We then have $\hat{c}_1 \hat{\alpha}_1 = (\mu_1 a + \mu_2 b) \hat{\alpha}_1 = a \cdot \widehat{\mu_1 \alpha_1} + b \cdot \widehat{\mu_2 \alpha_1} = a(1 - \mu_2 \alpha_2 - \mu_3 n) + b \cdot \widehat{\mu_2 \alpha_1} = \hat{a} + \hat{\mu}_2(b \alpha_1 - a \alpha_2)$.

From (5) we get that $t = \frac{a\alpha_2 - b\alpha_1}{\det M} = \frac{a\alpha_2 - b\alpha_1}{\pm n} \implies a\alpha_2 - b\alpha_1 = \pm n$. Consequently $\hat{c}_1 \hat{\alpha}_1 = \hat{a}$, q.e.d.

Similarly $\hat{c}_1 \hat{\alpha}_2 = (\mu_1 a + \mu_2 b) \hat{\alpha}_2 = a \cdot \widehat{\mu_1 \alpha_2} + b \cdot \widehat{\mu_2 \alpha_2} = a \cdot \widehat{\mu_1 \alpha_2} + b(1 - \mu_1 \alpha_1 - \mu_3 n) = \hat{b} + \hat{\mu}_1(a \alpha_2 - b \alpha_1) = \hat{b}$, q.e.d.

Similarly, by choosing $\hat{c}_2 = \widehat{\mu_1 A + \mu_2 B}$ one obtains the expressions of (6) regarding A, B .

We have obtained that (6) follows from the relations $\gcd(\alpha_1, \alpha_2, n) = 1$, (5) and $\det(M) = \pm n$. Moreover, we can renounce at the first equality of (5) since it can be deduced from the condition $\det(M) = \pm n$:

In did, if $\det(M) = \pm n$ we have $M \cdot M^* = \det(M) \cdot I_2 = (\pm n)I_2$, hence the existence of the elements α, β, u, v follows from the existence of the adjoint matrix M^* .

In conclusion, the hypothesis $E \simeq \frac{E}{C}$ with C cyclic subgroup of order n is equivalent with the following four relations: the second equality from (5), $(\alpha_1, \alpha_2, n) = 1$, $\det(M) = \pm n$ and $(a + b\tau)\tau = A + B\tau$.

We prove now that we can renounce at the second equality of (5) and at $\gcd(\alpha_1, \alpha_2, n) = 1$, by replacing them in the above equivalence with the condition $\gcd(a, A, b, B) = 1$:

Let $d = \gcd(a, A, b, B)$. From the second equality from (5) we obtain $d|\alpha_1, d|\alpha_2$ and since $d|\det M = \pm n$, it follows that $d|\gcd(\alpha_1, \alpha_2, n) = 1$, i.e. $\gcd(a, A, b, B) = 1$. We have that M is arithmetically equivalent with:

$$M = \begin{pmatrix} a & A \\ b & B \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}, \quad \text{i.e. } \exists U, V \in SL_2(\mathbb{Z}) \quad \text{such that}$$

$$\begin{pmatrix} a & A \\ b & B \end{pmatrix} = U \cdot \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \cdot V$$

The second condition of (5) becomes:

$$\begin{pmatrix} a & A \\ b & B \end{pmatrix} \cdot \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \iff U \cdot \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \cdot V \cdot \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

We denote by $V \cdot \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s' \\ t' \end{pmatrix}$ and obtain the equivalent equation

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = U \cdot \begin{pmatrix} s' \\ nt' \end{pmatrix}, U \in SL_2(\mathbb{Z}).$$

Viceversa, remark that the second condition of (5) follows from $\det(M) = \pm n$ and from the relation $\gcd(a, A, b, B) = 1$ as it follows: choose $s', t' \in \mathbb{Z}$ such that $(s', n) = 1$. Consider the matrix $(\alpha_1 \quad \alpha_2) = (s' \quad nt') \cdot U^t$, where U is an invertible matrix from the arithmetically equivalent decomposition of M . We have that $\gcd(\alpha_1, \alpha_2, n) = \gcd((\alpha_1, \alpha_2), n) = \gcd((s', nt'), n) =$

$\gcd(s', n) = 1$ and the second equality of (5) follows from the previous construction, q.e.d.

Let us analyze now the condition $(a + b\tau)\tau = A + B\tau$. This means that τ is algebraic over \mathbb{Q} and satisfies the second degree equation $b\tau^2 = (B - a)\tau + A$ (*).

Let $\mu_\tau = X^2 - uX - v \in \mathbb{Q}[X]$ be the minimal polynomial of τ , with $u = \frac{u_1}{u_2} \in \mathbb{Q}, v = \frac{v_1}{v_2} \in \mathbb{Q}, u_1, u_2, v_1, v_2 \in \mathbb{Z}, \gcd(u_1, u_2) = \gcd(v_1, v_2) = 1$ and let $d_2 := \gcd(u_2, v_2)$. We identify now the coefficients of the equation (*) and of the minimal polynomial of τ and obtain:

$$\begin{cases} \frac{A}{b} = v = \frac{v_1}{v_2} \\ \frac{B-a}{b} = u = \frac{u_1}{u_2} \end{cases} \quad \text{hence} \quad \begin{cases} v_2 A = v_1 b \\ u_2 (B - a) = u_1 b \end{cases}$$

$$\text{Since} \begin{cases} v_2 A = v_1 b \\ \gcd(v_1, v_2) = 1 \end{cases} \quad \text{we get that} \quad \begin{cases} v_2 | b \\ v_1 | A \end{cases}$$

$$\text{From} \begin{cases} u_2 (B - a) = u_1 b \\ \gcd(u_1, u_2) = 1 \end{cases} \quad \text{it follows that} \quad \begin{cases} u_2 | b \\ u_1 | B - a \\ \text{we already have} \end{cases} \quad \text{hence} \quad \text{lcm}[u_2, v_2] | b.$$

$$\text{Consequently} \quad \exists b' \in \mathbb{Z} \quad \text{such that} \quad \begin{cases} b = [u_2, v_2] b' = \frac{u_2 v_2}{d_2} b' \\ v_2 A = v_1 b \\ u_2 (B - a) = u_1 b \end{cases} \quad \text{hence} \quad \begin{cases} b = \frac{u_2 v_2}{d_2} b' \\ A = \frac{u_2 v_1}{d_2} b' \\ B = a + \frac{u_1 v_2}{d_2} b' \end{cases}$$

where $d_2 = \gcd(u_2, v_2)$ and $\tau^2 = u\tau + v$.

We prove now the equivalence:

$$\gcd(a, A, b, B) = 1 \iff \gcd(a, b') = 1.$$

" \implies " Denote by $d' = \gcd(a, b')$. It follows that $d' | \gcd(a, A, b, B)$ hence $d' | 1$, i.e. $d' = 1$.

" \impliedby " Let $d = \gcd(a, A, b, B)$. Then $d | a$ and since $\gcd(a, b') = 1$ we get $\gcd(d, b') = 1$. Moreover $d_2 = (u_2, v_2)$ so let $u_2 = d_2 u'_2$ and $v_2 = d_2 v'_2$ with $\gcd(u'_2, v'_2) = 1$. Since $d | a$ and $d | B$ we have that $d | B - a$ and obtain:

$$\begin{cases} d | b = \frac{u_2 v_2}{d_2} b' \\ d | A = \frac{u_2 v_1}{d_2} b' = u'_2 v_1 b' \\ d | B - a = \frac{u_1 v_2}{d_2} b' = v'_2 u_1 b' \end{cases} \quad \text{hence} \quad \begin{cases} d | \frac{u_2 v_2}{d_2} = u'_2 v_2 = u_2 v'_2 \\ d | u'_2 v_1 \\ d | v'_2 u_1 \end{cases}$$

If $d | u'_2$, since $\gcd(u'_2, v'_2) = 1$ one obtains $d | u_1$. But $\gcd(u_1, u_2) = 1$ and consequently $d = 1$.

If $d | v_2$, using the fact that $\gcd(v_1, v_2) = 1$ we get that $d | u'_2$. Since $d | u'_2$ and $\gcd(u_1, u_2) = 1$ we have $\gcd(d, u_1) = 1$. Consequently $d | v'_2$ and since we already have $d | u'_2$, we obtain $d = 1$.

We prove now that $n = \det(M) > 0$:

In order to simplify the notations, denote by $d := d_2$ and note that $(a, A, b, B) = \left(a, \frac{u_2 v_1}{d} b', \frac{u_2 v_2}{d} b', a + \frac{u_1 v_2}{d} b'\right)$.

We obtain that $\det(M) = aB - bA = a\left(a + \frac{u_1 v_2}{d} b'\right) - \left(\frac{u_2 v_2}{d} b'\right) \left(\frac{u_2 v_1}{d} b'\right) = a^2 + \frac{u_1 v_2}{d} a b' - \frac{v_1 v_2 u_2^2}{d^2} b'^2 = \left(a + \frac{u_1 v_2}{2d} b'\right)^2 - \frac{u_1^2 v_2^2}{4d^2} b'^2 - \frac{v_1 v_2 u_2^2}{d^2} b'^2 = \left(a + \frac{u_1 v_2}{2d} b'\right)^2 - \frac{u_1^2 v_2^2 + 4v_1 v_2 u_2^2}{4d^2} b'^2 = \left(a + \frac{u_1 v_2}{2d} b'\right)^2 - \frac{u_2^2 v_2^2 \left[\left(\frac{u_1}{u_2}\right)^2 + 4\frac{v_1}{v_2}\right]}{4d^2} b'^2 = \left(a + \frac{u_1 v_2}{2d} b'\right)^2 - \frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2$. Since $\Delta < 0$ and $u_2 \neq 0, v_2 \neq 0$ it follows easily that $\det(M) > 0$.

In order to describe the subgroup C we deduce that:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = U \cdot \begin{pmatrix} s' \\ nt' \end{pmatrix} \iff \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \cdot \begin{pmatrix} s' \\ nt' \end{pmatrix}$$

i.e. $\begin{cases} \alpha_1 = u_{11}s' + u_{12}nt' \\ \alpha_2 = u_{21}s' + u_{22}nt' \end{cases}$.

We obtain that:

$$g = \frac{\alpha_1 + \alpha_2 \tau}{n} = \frac{u_{11}s' + u_{12}nt' + (u_{21}s' + u_{22}nt')\tau}{n} = \frac{u_{11}s' + u_{21}s'\tau + n(u_{12}t' + u_{22}t'\tau)}{n} = \frac{s'(u_{11} + u_{21}\tau)}{n} + t'(u_{12} + u_{22}\tau) = g' + t'(u_{12} + u_{22}\tau),$$

where we denoted by $g' = \frac{s'(u_{11} + u_{21}\tau)}{n}$.

Consequently, $C = \frac{L+\mathbb{Z}g}{L} = \frac{L+\mathbb{Z}g'}{L} = \langle \overline{g'} \rangle$. But $\gcd(s', n) = \gcd(\alpha_1, \alpha_2, n) = 1$ hence there exist $a, b \in \mathbb{Z}$ such that $as' + bn = 1$. It follows that $ag' = a \frac{s'(u_{11} + u_{21}\tau)}{n} = \frac{(1-bn)(u_{11} + u_{21}\tau)}{n} = \frac{u_{11} + u_{21}\tau}{n} - b(u_{11} + u_{21}\tau)$. We have obtained that $C = \frac{L+\mathbb{Z}g'}{L} = \langle \overline{g'} \rangle = \langle \frac{u_{11} + u_{21}\tau}{n} \rangle$.

Remark 2.2. For a, b' fixed integers such that $\gcd(a, b') = 1$ the subgroup C is uniquely determined hence we can denote $C = C_{a, b'}$.

Recall now that for E, E' complex elliptic curves and $C \leq (E, +)$, $C' \leq (E', +)$ cyclic subgroups of order n of E and E' respectively, one has that $(E, C) \sim (E', C') \iff \exists u : E \rightarrow E'$ isomorphism such that $u(C) = C'$.

The question we want to answer is the following:

” Let E be a complex elliptic curve. When $\exists C \leq (E, +)$ cyclic subgroup of order n of E such that $(E, C) \sim (\frac{E}{C}, \frac{E[n]}{C})$, where $E[n] = \{x \in E : nx = 0\}$?”

In order to answer this question we will give the following criterion:

Theorem 2.3. *Let E be an elliptic curve defined over \mathbb{C} satisfying the conditions of 2.1 i). Then the following are equivalent:*

$$\begin{aligned} \{ \exists C \leq (E, +) \text{ cyclic subgroup of order } n \text{ of } E \text{ such that } (E, C) \sim (\frac{E}{C}, \frac{E[n]}{C}) \} &\iff \\ \{ \exists (a, b') \in \mathbb{Z}^2 \text{ with } \gcd(a, b') = 1 \text{ such that } \det(M) = n \text{ and } n | \text{Tr}(M) \} &\iff \\ \{ \exists (a, b') \in \mathbb{Z}^2, \text{ with } \gcd(a, b') = 1 \text{ such that } \det(M) = n \text{ and } M^2 \equiv O_2 \pmod{n} \} &\} \end{aligned}$$

where M is the matrix from 2.1 ii) and $\text{Tr}(M)$ the trace of M .

Proof. We use the notations of 2.1.

Let $E = \frac{\mathbb{C}}{L}$ with $L = \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$ lattice and $C = \langle \bar{g} \rangle \leq E = \frac{\mathbb{C}}{L}$ subgroup cyclic of order n , consequently $C = \frac{L+\mathbb{Z}g}{L}$. We have that $\frac{E}{C} \simeq \frac{\frac{\mathbb{C}}{L}}{\frac{L+\mathbb{Z}g}{L}} \simeq \frac{\mathbb{C}}{L+\mathbb{Z}g} = \frac{\mathbb{C}}{L'}$, where we denoted by $L' = L + \mathbb{Z}g = \mathbb{Z} + \mathbb{Z}\tau + \mathbb{Z}g$. Recall that $\frac{\mathbb{C}}{L} \simeq \frac{\mathbb{C}}{L'} \Leftrightarrow \exists \lambda \in \mathbb{C}$ such that $\lambda L = L'$.

Let u be defined in the following way: $E = \frac{\mathbb{C}}{L} \longrightarrow \frac{E}{C} = \frac{\mathbb{C}}{L'} = \frac{\mathbb{C}}{L+\mathbb{Z}g}$ isomorphism, $u(\bar{z}) = \widehat{\lambda z}$. Then $u(C) = u\left(\frac{L+\mathbb{Z}g}{L}\right) = \lambda \cdot \frac{L+\mathbb{Z}g}{L'} = \frac{\lambda L + \mathbb{Z}\lambda g}{L'}$. Since $\lambda L = L'$ we have that $u(C) = \frac{L' + \mathbb{Z}\lambda g}{L'} = \langle \widehat{\lambda g}, + \rangle$. But $E[n] = \frac{\frac{1}{n}L}{L}$ and consequently $\frac{E[n]}{C} = \frac{\frac{1}{n}L}{C} = \frac{\frac{1}{n}L}{\frac{L+\mathbb{Z}g}{L}} = \frac{\frac{1}{n}L}{L+\mathbb{Z}g} = \frac{\frac{1}{n}L}{L'}$. We obtain that:

$$(8) \quad u(C) = \frac{E[n]}{C} \Longleftrightarrow L' + \mathbb{Z}\lambda g = \frac{1}{n} \cdot L$$

The relation (7) is equivalent to $\lambda L + \mathbb{Z}\lambda g = \frac{1}{n} \cdot L \Longleftrightarrow \lambda(L + \mathbb{Z}g) = \frac{1}{n} \cdot L \Longleftrightarrow \lambda L' = \frac{1}{n} \cdot L \Longleftrightarrow \lambda^2 L = \frac{1}{n} \cdot L \Longleftrightarrow n\lambda^2 L = L$. Moreover,

$$(9) \quad L = n\lambda^2 L \Longleftrightarrow \exists M_1 \in SL_2(\mathbb{Z}) \quad \text{such that} \quad \begin{pmatrix} n\lambda^2 \\ n\lambda^2\tau \end{pmatrix} = M_1 \cdot \begin{pmatrix} 1 \\ \tau \end{pmatrix}$$

We denote by $M_1 = \begin{pmatrix} a_1 & A_1 \\ b_1 & B_1 \end{pmatrix}$ and by using the relations (3): $\begin{cases} \lambda = \frac{a+b\tau}{n} \\ \lambda\tau = \frac{A+B\tau}{n} \end{cases}$ we get:

$$\begin{aligned} \begin{pmatrix} n\lambda^2 \\ n\lambda^2\tau \end{pmatrix} &= \begin{pmatrix} a_1 & A_1 \\ b_1 & B_1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \tau \end{pmatrix} \Longleftrightarrow \\ \begin{cases} n\lambda^2 = a_1 + A_1\tau \\ n\lambda^2\tau = b_1 + B_1\tau \end{cases} &\Longleftrightarrow \begin{cases} a_1 + A_1\tau = \frac{(a+b\tau)^2}{n} \\ b_1 + B_1\tau = \lambda \cdot (A + B\tau) \end{cases} \Longleftrightarrow \begin{cases} a_1 + A_1\tau = \frac{(a+b\tau)^2}{n} \\ b_1 + B_1\tau = \frac{(a+b\tau)(A+B\tau)}{n} \end{cases} \end{aligned}$$

Note that $A + B\tau = \tau \cdot (a + b\tau)$, which implies that $b\tau^2 = (B - a)\tau + A$. By using this equation we obtain:

$$\begin{aligned} a_1 + A_1\tau &= \frac{(a+b\tau)^2}{n} \Longleftrightarrow na_1 + nA_1\tau = a^2 + b^2\tau^2 + 2ab\tau = a^2 + 2ab\tau + b[(B - a)\tau + A] = \\ &= (a^2 + bA) + b(a + B)\tau \Longleftrightarrow \\ &\begin{cases} a_1 = \frac{a^2+bA}{n} \\ A_1 = \frac{b(a+B)}{n} \end{cases} \\ b_1 + B_1\tau &= \frac{(a+b\tau)(A+B\tau)}{n} = \frac{\tau(a+b\tau)^2}{n} \Longleftrightarrow nb_1 + nB_1\tau = \tau[(a^2 + bA) + b(a + B)\tau] = \\ &= (a^2 + bA)\tau + (a + B)((B - a)\tau + A) = (a + B)A + (B^2 + bA)\tau \Longleftrightarrow \\ &\begin{cases} b_1 = \frac{(a+B)A}{n} \\ B_1 = \frac{B^2+bA}{n} \end{cases} \end{aligned}$$

We have that:

$$M^2 = \begin{pmatrix} a & A \\ b & B \end{pmatrix} \cdot \begin{pmatrix} a & A \\ b & B \end{pmatrix} = \begin{pmatrix} a^2 + bA & A(a + B) \\ b(a + B) & B^2 + bA \end{pmatrix} = \begin{pmatrix} na_1 & nb_1 \\ nA_1 & nB_1 \end{pmatrix}.$$

It is easy to see that the relation (8) is equivalent to $M^2 \equiv O_2(\text{mod } n)$:

" \implies " We proved above that $M^2 = n \cdot M_1^t \equiv O_2(\text{mod } n)$.

" \impliedby " Since $M^2 \equiv O_2(\text{mod } n)$ we obtain that there exist $a_1, b_1, A_1, B_1 \in \mathbb{Z}$ such that

$$M^2 = \begin{pmatrix} na_1 & nb_1 \\ nA_1 & nB_1 \end{pmatrix} = n \cdot M_1^t, \quad \text{where we denote by } M_1 = \begin{pmatrix} a_1 & A_1 \\ b_1 & B_1 \end{pmatrix}.$$

Since $M^2 = n \cdot M_1^t$ and $\det(M) = n$ we obtain that $n^2 = n^2 \cdot \det(M_1)$, hence $\det(M_1) = 1$, i.e. $M_1 \in SL_2(\mathbb{Z})$. From the previous proof we obtain that the relation (8) is satisfied, q.e.d.

Moreover, from Hamilton-Cayley's theorem we know that

$$M^2 - (\text{Tr}(M)) \cdot M + \det(M)I_2 = 0_2.$$

Since $\det(M) = n$, $M \in SL_2(\mathbb{Z})$ and $\gcd(a, A, b, B) = 1$ it follows easily that

$$M^2 \equiv O_2(\text{mod } n) \iff n | \text{Tr}(M)$$

which completes the proof. □

Proposition 2.4. *Let E be a complex elliptic curve satisfying the condition of Theorem 2.1 i), $\tau^2 = u\tau + v$, $u, v \in \mathbb{Q}$, $\Delta = u^2 + 4v < 0$ (i.e. E admits complex multiplication).*

$\exists C \leq (E, +)$ cyclic subgroup of order n of E such that $(E, C) \sim (\frac{E}{C}, \frac{E[n]}{C})$ only when $n \in \{1, 2, 3, \frac{-u_2^2 v_2^2 \Delta}{4d^2}, \frac{-u_2^2 v_2^2 \Delta}{d^2}\}$ and moreover, this is happening only when the following conditions are satisfied:

- a) *The case $n = \frac{-u_2^2 v_2^2 \Delta}{4d^2}$ occurs $\iff 2d | u_1 v_2$.
This case is realized for $b' = \pm 1$ and $a = -\frac{u_1 v_2}{2d} b'$.*
- b) *The case $n = \frac{-u_2^2 v_2^2 \Delta}{d^2}$ occurs $\iff 2d \nmid u_1 v_2$.
This case is realized for $b' = \pm 2$ and $a = -\frac{u_1 v_2}{2d} b'$.*
- c) *The case $n = 2$ occurs $\iff \frac{-u_2^2 v_2^2 \Delta}{4d^2} = 1$ and $2d | u_1 v_2$.
This case is realized for $b' = \pm 1$ and $a = \pm 1 - \frac{u_1 v_2}{2d} b'$.*
- d) *The case $n = 3$ occurs $\iff \frac{-u_2^2 v_2^2 \Delta}{d^2} = 3$ and $2d \nmid u_1 v_2$.
This case is realized for $b' = \pm 1$ and $a = \pm \frac{3}{2} - \frac{u_1 v_2}{2d} b'$.*

Remark 2.5. For each complex elliptic curve satisfying the condition of Theorem 2.1 i) there exists a unique $n \geq 2$ for which $\exists C \leq (E, +)$ cyclic subgroup of order n such that $(E, C) \sim (\frac{E}{C}, \frac{E[n]}{C})$. In other words, if for a given n there exist cyclic subgroups of order n of E such that $(E, C) \sim (\frac{E}{C}, \frac{E[n]}{C})$ then for a different integer m there are no cyclic subgroups of the same E such that $(E, C) \sim (\frac{E}{C}, \frac{E[m]}{C})$.

Proof. From the proof of Theorem 2.1 we have $(a, A, b, B) = \left(a, \frac{u_2 v_1}{d} b', \frac{u_2 v_2}{d} b', a + \frac{u_1 v_2}{d} b'\right)$

and $n = \left(a + \frac{u_1 v_2}{2d} b'\right)^2 - \frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2$.

From Theorem 2.3 we have that there exist $C \leq (E, +)$ cyclic subgroup of order n of E such that $(E, C) \sim \left(\frac{E}{C}, \frac{E[n]}{C}\right)$ if and only if $n = \det M$ and $n | \text{Tr}(M) = a + B$.

The relation $n | \text{Tr}(M) = a + B$ is equivalent to

$$(10) \quad n = \left(a + \frac{u_1 v_2}{2d} b'\right)^2 - \frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2 \quad | \quad 2 \left(a + \frac{u_1 v_2}{2d} b'\right)$$

By denoting $x = a + \frac{u_1 v_2}{2d} b'$, (9) becomes

$$(11) \quad x^2 - \frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2 \quad | \quad 2x$$

If $x > 2$ then $x^2 > 2x$ and consequently the equation (10) is impossible.

If $x < -2$ then $x^2 > -2x$ and consequently the equation (10) is impossible.

We obtain that $x \in [-2, 2]$, $x = \frac{m}{2}$, $m \in \mathbb{Z}$.

If $x = 2$, by using the fact that $\Delta < 0$, (10) becomes $4 - \frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2 \quad | \quad 4$, i.e. $b' = 0$. Consequently $x = a = 2$, $n = 4$ and $(a, b') = (2, 0)$. But $\gcd(a, b') = 1$, contradiction.

If $x = -2$ we obtain similarly that $b' = 0$ and $x = a = -2$, hence $\gcd(a, b') = 2$, contradiction.

If $x = 0$ we have that $a = -\frac{u_1 v_2}{2d} b'$. We distinguish two cases:

I If $2d | u_1 v_2$ it follows that $b' | a$. Since $\gcd(a, b') = 1$ it follows that $b' = \pm 1$.

Since $b' = \pm 1$ we have that $n = x^2 - \frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2 = -\frac{u_2^2 v_2^2 \Delta}{4d^2}$.

This case is realized for $b' = \pm 1$ and $a = -\frac{u_1 v_2}{2d} b'$.

II If $2d \nmid u_1 v_2$, by using the fact that $a = -\frac{u_1 v_2}{2d} b' \in \mathbb{Z}$, we have $2 | b' \implies \exists b'' \in \mathbb{Z}$ such that $b' = 2b''$. Consequently $a = -\frac{u_1 v_2}{d} b''$ and since $\gcd(a, b'') = 1$ we get that $b'' = \pm 1$ hence $b' = \pm 2$.

From $b' = \pm 2$ we obtain that $n = x^2 - \frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2 = -\frac{u_2^2 v_2^2 \Delta}{d^2}$.

This case is realized for $b' = \pm 2$ and $a = -\frac{u_1 v_2}{2d} b'$.

If $x = \pm 1$ then $n | 2x = a + B = \pm 2$, consequently $n \in \{1, 2\}$.

If $n = 1$ then C is trivial subgroup.

If $n = 2$ then $n = x^2 - \frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2 = 2 \iff -\frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2 = 1 \iff \frac{u_1^2 v_2^2 + 4v_1 v_2 u_2^2}{4d^2} b'^2 = -1$. (*)

If $2 | b'$ then we have the equation $\left(\frac{b'}{2}\right)^2 \cdot \left[u_1^2 \left(\frac{v_2}{d}\right)^2 + v_1 v_2 \left(\frac{u_2}{d}\right)^2\right] = -1$, hence $\frac{b'}{2} = \pm 1$, i.e. $b' = \pm 2$.

The equation (*) becomes $u_1^2 v_2^2 + 4v_1 v_2 u_2^2 = -d^2$ and, by using the fact that $u_2 = du'_2$ and $v_2 = d \cdot (v'_2)$ we obtain that $u_1^2 (v'_2)^2 + 4v_1 v_2 (u'_2)^2 = -1$. But a perfect square is congruent to either 0(mod4) or 1(mod4), hence from the previous equation $-1 \equiv 0(\text{mod}4)$ or $-1 \equiv 1(\text{mod}4)$ respectively, contradiction.

It remains that $2 \nmid b'$. Since $a = \pm 1 - \frac{u_1 v_2}{2d} b' \in \mathbb{Z}$ we get $2d | u_1 v_2$. The equation (*) becomes

$n = b'^2 \cdot \left[\left(\frac{u_1 v_2}{2d}\right)^2 + v_1 v_2 \left(\frac{u_2}{d}\right)^2\right] = -1$, consequently $b'^2 = 1$. In conclusion $n = 2$ implies

$2d | u_1 v_2$ and $\frac{u_2^2 v_2^2 \Delta}{4d^2} = -1$.

This case is realized for $b' = \pm 1$ and $a = \pm 1 - \frac{u_1 v_2}{2d} b'$.

If $x = \pm \frac{3}{2}$ we have that $n | 2x = a + B = \pm 3$, hence $n \in \{1, 3\}$.

If $n = 1$ then C is trivial subgroup.

If $n = 3$ we have that $n = x^2 - \frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2 = \frac{9}{4} - \frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2 |2x = \pm 3$. The divisibility occurs if and only if $\frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2 = -\frac{3}{4} \iff b'^2 \cdot \frac{u_1^2 v_2^2 + 4v_1 v_2 u_2^2}{d^2} = -3$. It follows that either $b'^2 = 1$ or $b'^2 = 3$. The case $b'^2 = 3$ is impossible. Consequently, it remains that $b'^2 = 1$ and $\frac{u_2^2 v_2^2 \Delta}{4d^2} = -3$.

Since $x = \pm \frac{3}{2}$ we obtain that $a = \pm \frac{3}{2} - \frac{u_1 v_2}{2d} b' \in \mathbb{Z}$, consequently $2d \nmid u_1 v_2$. This case is realized for $b' = \pm 1$ and $a = \pm \frac{3}{2} - \frac{u_1 v_2}{2d} b'$.

If $x = \pm \frac{1}{2}$ then $n | 2x = a + B = \pm 1$, consequently C is trivial subgroup.

Vice-versa, if $2d | u_1 v_2$ or $2d \nmid u_1 v_2$ one can easily check that there exists n with the properties specified in the Proposition.

The fact that such an $n > 1$ is unique is also easy to check:

If $2d | u_1 v_2$ and $\frac{-u_2^2 v_2^2 \Delta}{4d^2} > 1$, there exists C cyclic of order $n = \frac{-u_2^2 v_2^2 \Delta}{4d^2}$.

If $2d | u_1 v_2$ and $\frac{-u_2^2 v_2^2 \Delta}{4d^2} = 1$, there exists C cyclic of order 2.

If $2d \nmid u_1 v_2$ and $n = \frac{-u_2^2 v_2^2 \Delta}{d^2} > 1$, there exists C cyclic of order $n = \frac{-u_2^2 v_2^2 \Delta}{d^2}$.

If $2d \nmid u_1 v_2$ and $n = \frac{-u_2^2 v_2^2 \Delta}{d^2} = 1$ we have that $\frac{u_1^2 v_2^2 + 4v_1 v_2 u_2^2}{d^2} = -1$. Since $d | u_2$ and $d | v_2$ it follows that there exist $u'_2, v'_2 \in \mathbb{Z}$ such that $u_2 = du'_2$ and $v_2 = dv'_2$. The equality $\frac{u_1^2 v_2^2 + 4v_1 v_2 u_2^2}{d^2} = -1$ means $u_1^2 (v'_2)^2 + 4v_1 v_2 (u'_2)^2 = -1$, i.e. $(u_1 (v'_2))^2 \equiv -1 \pmod{4}$, contradiction. Consequently, this case is not possible. \square

3. EXAMPLES. THE CASES $n = 2$, $n = 3$ AND $n = 5$

In this section we classify (up to an isomorphism) the elliptic curves E which admit a subgroup $C \leq (E, +)$ of order either 2 or 3 or 5 such that $\frac{E}{C} \simeq E$. We recall that complex elliptic curves are of the form $\frac{\mathbb{C}}{L}$ for some $L = \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$ where $\tau \in G = \left\{ z = x + iy : -\frac{1}{2} \leq x < \frac{1}{2}, |z| \geq 1 \right\}$.

Let E be an elliptic curve satisfying the condition of Theorem 2.1,i).

E is isomorphic to an elliptic curve $E' = \frac{\mathbb{C}}{L}$, where $L = \mathbb{Z} + \mathbb{Z}\tau$ and $\tau \in G$. Since an isomorphism $u : E \rightarrow E'$ is of the type $u(z) = A \cdot z, A \in SL_2(\mathbb{Z})$ one easily obtains that E' satisfies the condition of Theorem 2.1,i). Hence we can assume (up to an isomorphism) that E is of the form $\frac{\mathbb{C}}{L}$ with $L = \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$ and $\tau \in G$. Moreover, we observe that if $\tau = x + iy \in G$, from $|x| \leq \frac{1}{2}$ and $|z| = x^2 + y^2 \geq 1$ it follows that $y^2 \geq \frac{3}{4}$.

Let $\tau^2 - u\tau - v = 0, u, v \in \mathbb{Q}, \Delta = u^2 + 4v < 0$ and $\tau \in G$. Then $\tau = \frac{u \pm i\sqrt{|\Delta|}}{2}$ and, since $\tau \in G$, we get that $-1 \leq u < 1$ and $|\Delta| \geq 3$.

Since $\Delta = u^2 + 4v < 0$ we have that $v < 0$. Without loss of generality, we can assume that $v_2 > 0, v_1 < 0$ and $u_2 > 0$. By using Theorem 2.1 and these restrictions imposed to τ we obtain the following results:

Proposition 3.1. *There are exactly 3 elliptic curves E (up to an isomorphism) which admit at least one subgroup $C \leq (E, +)$ of order 2 such that $\frac{E}{C} \simeq E$.*

If we denote by $L = \mathbb{Z} + \mathbb{Z}\tau$, they are:

- a) $E = \frac{\mathbb{C}}{L}, \tau^2 = -1;$
- b) $E = \frac{\mathbb{C}}{L}, \tau^2 = -2;$
- c) $E = \frac{\mathbb{C}}{L}, \tau^2 = -\tau - 2.$

Remark 3.2. The Fricke involution w_2 of $Y_0(2)$ has 2 fixed points and they correspond to the cases a) and b); note that $\nu(2) = 2$.

Proof. From Theorem 2.1 we have:

$$(12) \quad 2 = aB - bA = \left(a + \frac{u_1 v_2}{2d} b'\right)^2 - \frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2$$

Since $\Delta \leq -3$ we obtain $2 = \left(a + \frac{u_1 v_2}{2d} b'\right)^2 - \frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2 \geq \left(a + \frac{u_1 v_2}{2d} b'\right)^2 + \frac{3u_2^2 v_2^2}{4d^2} b'^2 \geq \frac{3u_2^2 v_2^2}{4d^2} b'^2 \geq \frac{3b'^2}{4}$, hence $b'^2 \leq \frac{8}{3}$. Since $b' \in \mathbb{Z}$ we get that $b' \in \{0, \pm 1\}$.

If $b' = 0$ the equation (10) becomes $2 = a^2$, absurd.

If $b' = \pm 1$, the equation (10) leads to $2 \geq \frac{3}{4} \cdot \left(\frac{u_2 v_2}{d}\right)^2$.

Denote by $\frac{u_2 v_2}{d} = y \in \mathbb{Z}$. From the above inequality we get that $|y| \leq 1$.

I) If $y = 0$ then $u_2 v_2 = 0$, absurd.

II) If $y = \pm 1$ then $\frac{u_2 v_2}{d} = \pm 1$.

Since $u_2 > 0$ and $v_2 > 0$ we have $\frac{u_2 v_2}{d} = 1 \iff u_2 \cdot \frac{v_2}{d} = 1 \iff v_2 \cdot \frac{u_2}{d} = 1$. We obtain that $u_2 = \frac{v_2}{d} = 1$ and $v_2 = \frac{u_2}{d} = 1$, consequently $u_2 = v_2 = d = 1$. The equation (10) becomes:

$$(13) \quad 2 = \left(a + \frac{u_1}{2} b'\right)^2 - \frac{\Delta}{4}$$

Consequently $\tau^2 - u_1 \tau - v_1 = 0$, $u_1, v_1 \in \mathbb{Z}$, $\Delta = u_1^2 + 4v_1 < 0$. From $\tau \in G$ and $\tau = \frac{u_1 \pm i\sqrt{|\Delta|}}{2}$ we get that $-1 \leq u_1 < 1$. Since $u_1 \in \mathbb{Z}$ we have that $u_1 \in \{-1, 0\}$. We distinguish the following cases:

a) $u_1 = 0$.

The equation (12) becomes $2 = a^2 - \frac{\Delta}{4} = a^2 - \frac{4v_1}{4} = a^2 - v_1$. But $v_1 < 0$ and $v_1 \in \mathbb{Z}$, consequently $a^2 = -v_1 = 1$ sau $a^2 = 0$, $-v_1 = 2$.

If $a^2 = -v_1 = 1$ it follows that $\tau^2 = -1$ and $(a, b') = (\pm 1, \pm 1)$ (hence a and b' are co-primes).

If $a^2 = 0$, $-v_1 = 2$ we obtain that $\tau^2 = -2$ and $(a, b') = (0, \pm 1)$ (hence a and b' are co-primes).

b) $u_1 = -1$.

The equation (12) becomes $2 = \left(a - \frac{b'}{2}\right)^2 - \frac{\Delta}{4} = \left(a - \frac{b'}{2}\right)^2 - \frac{1+4v_1}{4}$.

Since $b' = \pm 1$ we distinguish two cases:

b1) If $b' = 1$ we have that $2 = \left(a - \frac{1}{2}\right)^2 - \frac{1+4v_1}{4} \iff 8 = (2a-1)^2 - 1 - 4v_1 \iff 9 = (2a-1)^2 - 4v_1$.

On the other hand $v_1 \leq -1$ hence the only possibility for the previous equation is $v_1 = -2$ and $2a-1 = \pm 1$. From $u_1 = -1$ and $v_1 = -2$ it follows that τ satisfies the equation $\tau^2 + \tau + 2 = 0$. Moreover, if $2a-1 = 1$ it follows that $(a, b') = (1, 1)$ (a and b' are co-primes).

If $2a-1 = -1$ it follows that $(a, b') = (0, 1)$ (a and b' are co-primes).

b2) If $b' = -1$ we have that $2 = \left(a + \frac{1}{2}\right)^2 - \frac{1+4v_1}{4} \iff 8 = (2a+1)^2 - 1 - 4v_1 \iff 9 = (2a+1)^2 - 4v_1$.

Since $v_1 \leq -1$ we obtain similarly that $v_1 = -2$ and $2a+1 = \pm 1$. Since $u_1 = -1$ and $v_1 = -2$ we obtain that τ verifies the same equation $\tau^2 + \tau + 2 = 0$. \square

Remark 3.3. In particular, given a complex elliptic curve E in the form $\mathbb{C}/\langle 1, \tau \rangle$ such that $\tau \in G$, there are exactly three values of τ for which E admits an endomorphism of degree 2:

$\tau = i; \tau = \sqrt{-2}; \tau = \frac{-1+\sqrt{-7}}{2}$, (see [Ha], Exercise 4.12), emphasizing in this way the validity of our results.

Proposition 3.4. *There are exactly 4 elliptic curves E (up to an isomorphism) which admit at least one subgroup $C \leq (E, +)$ of order 3 such that $\frac{E}{C} \simeq E$. If we put $L = \mathbb{Z} + \mathbb{Z}\tau$, they are:*

- a) $E = \frac{\mathbb{C}}{L}, \tau^2 = -2;$
- b) $E = \frac{\mathbb{C}}{L}, \tau^2 = -3;$
- c) $E = \frac{\mathbb{C}}{L}, \tau^2 = -\tau - 1;$
- d) $E = \frac{\mathbb{C}}{L}, \tau^2 = -\tau - 3.$

Remark 3.5. The Fricke involution w_3 of $Y_0(3)$ has 2 fixed points and they correspond to the cases b) and c); note that $\nu(3) = 2$.

Proof. By using Theorem 2.1,i) we have that:

$$(14) \quad 3 = aB - bA = \left(a + \frac{u_1 v_2}{2d} b'\right)^2 - \frac{u_2^2 v_2^2 \Delta}{4d^2} b'^2$$

Since $|\Delta| \geq 3$ we obtain that $3 \geq \frac{3u_2^2 v_2^2}{4d^2} b'^2 \geq \frac{3}{4} b'^2$. Consequently $b'^2 \leq 4$.

I) If $b'^2 = 4$ both sides of the previous inequality are equal since $3 = \frac{3}{4} b'^2$, hence $\Delta = -3$ and $\frac{u_2 v_2}{d} = 1$. Since $u_2 > 0$ and $v_2 > 0$ we get that $u_2 = v_2 = d = 1$. Consequently τ satisfies the equation $\tau^2 - u_1 \tau - v_1 = 0, u_1, v_1 \in \mathbb{Z}, \Delta = u_1^2 + 4v_1 < 0$. From $\tau \in G$ and $\tau = \frac{u_1 \pm i\sqrt{|\Delta|}}{2}$ we obtain that $-1 \leq u_1 < 1$. Since $u_1 \in \mathbb{Z}$ we get that $u_1 \in \{-1, 0\}$. By using the equation $\Delta = -3$ we obtain that $u_1^2 + 4v_1 = -3$, consequently u_1 is odd. It follows that $u_1 = -1$ and $v_1 = -1$, consequently τ satisfies $\tau^2 + \tau + 1 = 0$.

II) If $b'^2 = 1$ by using the equation (14) we get that $3 \geq \frac{3}{4} \cdot \left(\frac{u_2 v_2}{d}\right)^2$.

We denote by $\frac{u_2 v_2}{d} = y \in \mathbb{Z}$. Since $u_2 > 0$ and $v_2 > 0$ we have that $y > 0$. From the above inequality we obtain that $y \in \{1, 2\}$.

a) If $y = 1$ we have that $u_2 = v_2 = d = 1$.

Consequently τ satisfies the equation $\tau^2 - u_1 \tau - v_1 = 0, u_1, v_1 \in \mathbb{Z}, \Delta = u_1^2 + 4v_1 < 0$.

Since $\tau \in G$ and $\tau = \frac{u_1 \pm i\sqrt{|\Delta|}}{2}$ we get that $-1 \leq u_1 < 1$. Since $u_1 \in \mathbb{Z}$ it follows that $u_1 \in \{-1, 0\}$. We distinguish two cases:

a1) If $u_1 = 0$ the equation (14) becomes $3 = a^2 - \frac{1}{4}\Delta = a^2 - v_1$. On the other hand $v_1 < 0$ and $v_1 \in \mathbb{Z}$, consequently either $a^2 = 1, v_1 = -2$ or $a^2 = 0, v_1 = -3$.

If $a^2 = 1, v_1 = -2$ we have that $u_1 = 0$ and $v_1 = -2$. It follows that τ satisfies the equation $\tau^2 = -2$.

If $a^2 = 0, v_1 = -3$ we have that $u_1 = 0$ and $v_1 = -3$. It follows that τ satisfies the equation $\tau^2 = -3$.

a2) If $u_1 = -1$ the equation (14) becomes $3 = \left(a - \frac{b'}{2}\right)^2 - \frac{1}{4}\Delta = \left(a - \frac{b'}{2}\right)^2 - \frac{1}{4} \cdot (1 + 4v_1) \iff 13 = (2a - b')^2 - 4 \cdot v_1$. From $b'^2 = 1$ we get that $2a - b'$ is odd. Since $v_1 \in \mathbb{Z}$ and $v_1 \leq -1$ we obtain that $13 \geq (2a - b')^2 + 4$ and consequently $2a - b' = \pm 1$ and $v_1 = -3$. It follows that τ satisfies the equation $\tau^2 + \tau + 3 = 0$ and $(a, b') \in \{(0, \pm 1), (0, -1), (-1, -1)\}$ (hence a and b' are co-primes).

b) If $y = 2$ we have that $\frac{u_2 v_2}{d} = 2$ hence $u_2 \cdot \frac{v_2}{d} = v_2 \cdot \frac{u_2}{d} = 2$.

Since $u_2 > 0$ and $v_2 > 0$ we get that $(u_2, v_2) \in \{(2, 1), (1, 2), (2, 2)\}$. We distinguish 3 cases:

b1) If $(u_2, v_2) = (2, 1)$ the equation (14) becomes $3 = \left(a + \frac{u_1 b'}{2}\right)^2 - \Delta$.

Since $\Delta \leq -3$ we get that $a + \frac{u_1 b'}{2} = 0$ and $\Delta = -3$. Since $a + \frac{u_1 b'}{2} = 0$ and $b' = \pm 1$ we obtain that u_1 is even. We have that $\tau^2 - \frac{u_1}{2}\tau - v_1 = 0$, $\tau \in G$ and $-1 \leq \frac{u_1}{2} < 1$. Consequently $-2 \leq u_1 < 2$ and, since u_1 is even, it follows that $u_1 \in \{-2, 0\}$.

If $u_1 = -2$, since $\Delta = -3$ we get that $\frac{u_1^2}{4} + 4v_1 = -3 \iff v_1 = -1$. In consequence τ satisfies the equation $\tau^2 + \tau + 1 = 0$.

If $u_1 = 0$, from $\Delta = -3$ we get that $\frac{u_1^2}{4} + 4v_1 = -3 \iff v_1 = -\frac{3}{4}$, contradiction.

b2) If $(u_2, v_2) = (1, 2)$ the equation (14) becomes $3 = (a + u_1 b')^2 - \Delta$.

Since $\Delta \leq -3$ we get that $a + u_1 b' = 0$ and $\Delta = -3$. We have that $\tau^2 - u_1 \tau - \frac{v_1}{2} = 0$, $\tau \in G$ and $-1 \leq u_1 < 1$ hence $u_1 \in \{-1, 0\}$.

If $u_1 = -1$, from $\Delta = -3$ we get that $u_1^2 + 2v_1 = -3 \iff v_1 = -2$. Consequently τ satisfies the equation $\tau^2 + \tau + 1 = 0$.

If $u_1 = 0$, from $\Delta = -3$ we get that $u_1^2 + 2v_1 = -3 \iff 2v_1 = -3$, contradiction.

b3) If $(u_2, v_2) = (2, 2)$ the equation (14) becomes $3 = \left(a + \frac{u_1 b'}{2}\right)^2 - \Delta$.

Since $\Delta \leq -3$ we obtain $a + \frac{u_1 b'}{2} = 0$ and $\Delta = -3$. Since $a + \frac{u_1 b'}{2} = 0$ and $b' = \pm 1$ we get that u_1 is even. From $\tau \in G$ and $\tau^2 - \frac{u_1}{2}\tau - \frac{v_1}{2} = 0$ we obtain that $-2 \leq u_1 < 2$. Since u_1 is even we get that $u_1 \in \{-2, 0\}$.

If $u_1 = -2$, from $\Delta = -3$ it follows that $\frac{u_1^2}{4} + 2v_1 = -3 \iff v_1 = -2$. In conclusion τ satisfies the equation $\tau^2 + \tau + 1 = 0$.

If $u_1 = 0$, from $\Delta = -3$ we obtain that $\frac{u_1^2}{4} + 2v_1 = -3 \iff v_1 = -\frac{3}{2}$, contradiction.

III) If $b'^2 = 0$ the equation (14) leads to $3 = a^2$, contradiction. \square

We also obtain the following:

Proposition 3.6. *There are exactly 6 elliptic curves E (up to an isomorphism) which admit at least one subgroup $C \leq (E, +)$ of order 5 such that $\frac{E}{C} \simeq E$. If we put $L = \mathbb{Z} + \mathbb{Z}\tau$, they are:*

- a) $E = \frac{\mathbb{C}}{L}, \tau^2 = -1$;
- b) $E = \frac{\mathbb{C}}{L}, \tau^2 = -4$;
- c) $E = \frac{\mathbb{C}}{L}, \tau^2 = -5$;
- d) $E = \frac{\mathbb{C}}{L}, \tau^2 = -\tau - 3$;
- e) $E = \frac{\mathbb{C}}{L}, \tau^2 = -\tau - 5$;
- f) $E = \frac{\mathbb{C}}{L}, \tau^2 = -\tau - \frac{3}{2}$.

Remark 3.7. The Fricke involution w_5 of $Y_0(5)$ has 2 fixed points and they correspond to the cases c) and f); note that $\nu(5) = h(-20) = 2$.

Proof. We use the same reasoning as in the Propositions 3.1 and 3.4 hence the proof is left to the reader as an exercise. \square

Remark 3.8. Given a prime number p , there are exactly $p + 1$ complex elliptic curves E (up to an isomorphism) which admit at least one subgroup $C \leq (E, +)$ of order p such that $\frac{E}{C} \simeq E$. Note also that given a prime number p , there are $p + 1$ unramified coverings of degree p of a (complex) elliptic curve.

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